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Concave programming and DH-point

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Abstract An extreme point property of optimal solutions of general concave programming problems is established that generalizes both Du-Hwang's minimax theorem and its continuous version by Du and Pardalos.

Keywords Steiner ratio · Du-Hwang minimax theorem · DH-point · Du-Pardalos' continuous version · General concave programming · DC programming · Optimality condition

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1 Introduction

A long standing conjecture of Gilbert and Pollak about the Steiner ratio was finally proved in 1990 by Du and Hwang [2] by using the following special "minimax theorem":

Let C be a polytope in $\mathbb{R}^n$ defined by the inequalities $\langle a^j, x\rangle \geq b_j,\ j = 1,\ldots,k$, and let $g_i(x),\ i = 1\ldots,m$, be continuous concave functions on $\mathbb{R}^n$. Then an optimal solution of the problem

$$\min\{\max_{i=1,\ldots,m} g_i(x)|\ x \in C\} \tag{1}$$

exists which is a DH-point, i.e., a point x^* such that if $J(x) := \{j|\ \langle a^j, x\rangle = b_j\},\ M(x) := \{i|\ g_i(x) = \max_{i'=1,\ldots,m} g_{i'}(x)\}$ then there is no $x \in C$ with $J(x) \supset J(x^*),\ M(x) \supset M(x^*)$ and $|J(x)| + |M(x)| > |J(x^*)| + |M(x^*)|$.

Subsequently, a continuous version of this proposition is established [3] that uses, however, a slightly modified (and weaker) concept of DH-point, and so does not actually generalize Du-Hwang's proposition but only a weaker version of it. The situation looks somewhat intriguing, because, despite its name and the fact that it deals with a property of a minimax

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problem, Du-Hwang's proposition has little to do with classical minimax theory as has been developed through the years since the work of von Neumann in 1928 (see e.g. [9]).

In fact, DH-property is rather related to the well known property that the minimum of a concave function over a compact convex set is attained at some extreme point of this compact convex set. The purpose of the present note is to show that, more precisely, DH-property is merely a special case of an extreme point property of concave minimization problems whose feasible set is a compact set formed by the solutions of a finite or infinite system of convex and reverse convex inequalities [8].

First we consider, in Sect. 2, the most important case when the reverse convex inequalities have the form $g(x, y) \geq 0, y \in D$, with $g(x, y)$ concave functions in x for every fixed $y \in D$. The class of such problems, to be referred to as general concave programming problems (GCP), includes essentially the class of DC optimization problems introduced and studied extensively during the last two decades (see e.g. [7,8]). In this section we state and prove the basic property of (GCP) that an optimal solution of it always exists which is an extreme point of the compact feasible set. Next, in Sect. 3, we introduce a concept of DH-point of a compact set which generalizes the original concept by Du-Hwang, and, under the assumption that $|D|$ (number of reverse convex constraints) is finite, we prove that any extreme point of the feasible set of (GCP) is a DH-point. Finally, in Sect. 4, we prove the existence of an optimal DH-point for any (GCP), where D is an arbitrary compact subset of a metric space Y and $g(x, y)$ is a continuous function on $C \times Y$, quasiconcave in x for every fixed y. As prerequisite, we assume that the reader is familiar with properties of extremal sets and extreme points of compact sets which can be found e.g. in [1,2], or also in [8], especially for faces and extreme points of compact convex sets.

2 General concave programming

By *general concave programming* problems we mean the class of optimization problems of the form

$$\min\{f(x)|\ \max_{y\in D} g(x, y) \leq 0, x \in C\}, \tag{GCP}$$

where C is a nonempty compact convex set in $\mathbb{R}^n$, $f(x)$ a continuous concave function on $C, \mathcal{D}$ a compact subset of a metric space Y, and $g(x, y)$ a continuous concave function in x for fixed $y \in D$. A subclass of this class of problems is constituted by DC optimization problems over compact convex sets, since these, as shown e.g. in [8], can always be reduced to the canonical form

$$\min\{f(x)|\ g(x) \leq 0,\ x \in C\},$$

where $f(x), g(x)$ are concave functions and C is a compact convex set.

On the other hand, when $D = \{1, \ldots, m\}$ and $g_i(x) := g(x, i), i = 1, \ldots, m$, as in the problem (1), the function $g(x) = \max_{i=1,\ldots,m} g_i(x)$ is a DC function, namely: $g(x) = u(x) - v(x)$, where

$$u(x) := \sum_{i=1}^{m} g_i(x), \quad v(x) := \min_{k=1,\ldots,m} \sum_{i\neq k} g_i(x)$$

are concave functions (see e.g. [8]). So in this case (GCP) can be written as the DC optimization problem

$$\min\{f(x)|\ u(x) - v(x) \leq 0,\ x \in C\}. \tag{2}$$

In particular, when $D = \emptyset$, (GCP) is reduced to the classical concave programming problem [8]:

$$\min\{f(x)|\, x \in C\}. \tag{CCP}$$

It is well known that in the latter particular case an optimal solution x^* always exists which is an extreme point of the compact convex set C. To generalize this property let us recall some definitions (see e.g. [1,2]).

A subset S of a compact set K is called an *extremal set* of K if for any $x', x'' \in K$, whenever $x = \alpha x' + (1-\alpha)x'' \in S$ for some $\alpha \in (0, 1)$ then $x', x'' \in S$. An extremal set of a compact convex set is also called a *face* of it. An extremal set which is a singleton is called an *extreme point*. The following result is known (see e.g., [5], Sect. 13A).

Theorem 1 *The minimum of a concave function $f(x)$ over a compact set K is achieved at an extreme point of K.*

Proof Let $\overline{\text{co}}K$ denote the closed convex hull of K. It is well known that the minimum of $f(x)$ over $\overline{\text{co}}K$ is achieved at an extreme point x^* of $\overline{\text{co}}K$, and that every extreme point of $\overline{\text{co}}K$ belongs to K, hence is an extreme point of K (Krein-Milman theorem, or Proposition 1.17 in [8]). The result follows, by observing that if x^* is an extreme point of $\overline{\text{co}}K$ which is a minimizer of $f(x)$ over $\overline{\text{co}}K$ then x^* is an extreme point of K and a minimizer of $f(x)$ over K.

An alternative proof of this Theorem can also be found in [5]. □

Letting

$$K = \{x \in \mathbb{R}^n|\ \max_{y \in D} g(x, y) \leq 0,\ x \in C\}, \tag{3}$$

we thus obtain:

Corollary 1 *At least an optimal solution of (GCP) is an extreme point of of the feasible set.*

3 Extreme point and DH-point — case $|D| < +\infty$

Given the above defined set K, for each $x \in C$ let $M(x) = \{y \in D|\, g(x, y) = 0\}$ and denote by $S(x)$ the smallest face of C that contains x. As is well known (see e.g., [8], Corollary 1.11), if $x' \in S(x) \setminus \text{ri}S(x)$ then $S(x') \subset S(x)$, and $S(x) \setminus S(x') \neq \emptyset$. In the case C is a polytope, e.g. $C = \{x|\ \langle a^j, x\rangle \geq b_j,\ j = 1, \ldots, k\}$, if $J(x) = \{j|\ \langle a^j, x\rangle = b_j\}$ then clearly for any $x' \in S(x) : S(x) \setminus S(x') \neq \emptyset$ if and only if $|J(x')| > |J(x)|$.

A point $x^* \in K$ is called a DH-*point* of K if there is no $x \in S(x^*) \cap K$ such that $M(x^*) \subset M(x)$ and: either $x \notin \text{ri}S(x^*)$ or $M(x) \setminus M(x^*) \neq \emptyset$. This condition amounts to requiring that there is no $x \in K$ such that

$$S(x) \subset S(x^*),\ M(x^*) \subset M(x),\ \text{and } (S(x^*) \setminus S(x)) \cup (M(x) \setminus M(x^*)) \neq \emptyset,$$

where we write $A \cup B \neq \emptyset$ to mean that at least one of the two sets A, B is nonempty. It is easily seen that a DH-point for problem (1) as defined in [3] is nothing but a DH-point of the set $\{x \in R^n|\ \max_{i=1,\ldots,m} g_i(x) \leq 0,\ x \in C\}$ in our definition.

Theorem 2 *If $|D|$ is finite, any extreme point x^* of K is a DH-point of K.*

Proof Let x^* be an extreme point of K and suppose there is $\hat{x} \in S(x^*) \cap K$ such that $M(x^*) \subset M(\hat{x})$ and $(S(x^*) \setminus S(\hat{x})) \cup (M(\hat{x}) \setminus M(x^*)) \neq \emptyset$. Then $\hat{x} \neq x^*$, and since $S(x^*)$ is the smallest face of C containing x^*, if a point x^λ is such that $x^* = (1-\lambda)x^\lambda + \lambda\hat{x} \in S(x^*)$ then $x^\lambda \in S(x^*)$ for all $\lambda > 0$ small enough. For every $y \in M(x^*) \subset M(\hat{x})$, by concavity of $g(., y)$, we have $0 = g(x^*, y) \geq (1-\lambda)g(x^\lambda, y) + \lambda g(\hat{x}, y) = (1-\lambda)g(x^\lambda, y)$, so $g(x^\lambda, y) \leq 0 \, \forall \lambda > 0$. On the other hand, if $y \in D \setminus M(x^*)$ then $g(x^*, y) < 0$, hence, noting that the function $x \mapsto \max_{y \in D\setminus M(x^*)} g(x, y)$ is continuous (because D is finite), we have $\max_{y \in \setminus M(x^*)} g(x^\lambda, y) < 0$ for small enough $\lambda > 0$. For these values of λ, $\max_{y \in D} g(x^\lambda, y) \leq 0$, i.e., $x^\lambda \in K$, conflicting with x^* being an extreme point. Therefore, x^* must be a DH-point. □

Corollary 2 *If (GCP) has only finitely many nonconvex constraints, at least an optimal solution of it is achieved at a DH-point of the feasible set.*

Proof This follows from Theorem 2 and Corollary 1. □

The minimax problem (1) considered in Du-Hwang [2] is a special case of (GCP), since it can be written as

$$\min\{t| \max_{i=1,\ldots,m} g_i(x) - t \leq 0, \; x \in C\}. \tag{4}$$

More generally, if $g(x) := \max_{y \in D} g(x, y)$, where D is an arbitrary set, the problem $\min\{g(x)| \; x \in C\}$ is equivalent to the special (GCP):

$$\min\{t| \max_{y \in D} g(x, y) \leq t, \; x \in C\}. \tag{5}$$

Lemma 1 *A point $(x^*, g(x^*))$ with $x^* \in C$ is a DH-point of the set $H = \{(x, t)| \max_{y \in D} g(x, y) \leq t, \; x \in C\}$ if and only if there is no $x \in S(x^*)$ such that $M(x) \supset M(x^*)$ and $|(S(x^*) \setminus S(x))| + |M(x) \setminus M(x^*))| > 0$, where $M(x) := \{y \in D| \; g(x, y) = g(x)\}$. In other words, $(x^*, g(x^*))$ is a DH-point of the set $H = \{(x, t)| \max_{y \in D} g(x, y) \leq t, \; x \in C\}$ if and only if x^* is a DH-point of the set $K = \{x \in C| \max_{y \in D} g(x, y) \leq g(x)\}$.*

Proof Clearly $(x^*, g(x^*)) \in H$. For $(x, t) \in H$, since $t \geq g(x) := \max_{y \in D} g(x, y)$, we have $g(x, y) - t = 0 \Leftrightarrow g(x, y) \geq g(x) \Leftrightarrow g(x, y) = g(x)$, so $M(x, t) := \{y \in D| \; g(x, y) = t\} = M(x)$. Hence, $(x^*, g(x^*))$ is a DH-point of H if and only if there is no $x \in S(x^*)$, such that $M(x) \supset M(x^*)$ and $(S(x^*) \setminus S(x)) \cup (M(x) \setminus M(x^*)) \neq \emptyset$. □

Since, by Corollary 2, an optimal solution $(x^*, g(x^*))$ of problem (4) exists which is a DH-point of the set $H = \{(x, t)| \max_{i \in I}[g_i(x) - t] \leq 0, \; x \in C\}$, we obtain from Lemma 1 Du-Hwang's result mentioned in the Introduction:

Corollary 3 (Du-Hwang [3]) *If $|D| < +\infty$ and $g(x, y), \; y \in D$, are concave functions, the minimum of $g(x) := \max_{y \in D} g(x, y)$ over a compact convex set C is achieved at a DH-point x^* of the set $K := \{x \in C| \max_{y \in D} g(x, y) \leq g(x)\}$.*

Thus, Du-Hwang's "minimax theorem" is merely a consequence of Corollary 2 and Lemma 1.

Remark 1 As we saw, when $D = \emptyset$, (GCP) reduces to the classical concave programming problem (CCP). If $C = \{x \in \mathbb{R}^n| \max_{z \in Z} h(x, z) \leq 0\}$, where $h(x, z)$ is a continuous convex function in x for every fixed z, the problem can be written as

$$\min\{f(x)| \max_{z \in Z} h(x, z) \leq 0\}.$$

Letting $N(x) = \{z \in Z|\ h(x,z) = 0\}$, it is not hard to see that for any $x^* \in C$ the set $S(x^*) = \{x|\ h(x,z) = 0\ \forall z \in N(x^*)\}$ is an extremal set of C. Therefore, x^* is an extreme point of C if and only if $S(x^*)$ is a singleton, i.e., if and only if there is no x such that $N(x^*) \subset N(x)$ and $N(x) \setminus N(x^*) \neq \emptyset$. In other words, *$x^*$ is an extreme point of a compact convex set C if and only if it is a DH-point of this set*. Based on this trivial property of extreme points of compact convex sets, an alternative proof of Corollary 2 is to reduce problem (1) to the following (CCP):

$$\min\{u(x) - t|\ \min_{i \in I} v_i(x) \geq t,\ c \in C\}, \tag{6}$$

where $u(x) = \sum_{i \in I} g_i(x),\ v(x) = \min_{i \in I} v_i(x)$, and $v_i(x) := \sum_{k \in I \setminus \{i\}} g_k(x)$ are concave functions (see (2)). In fact, an optimal solution $(x^*, v(x^*))$ of (6) exists which is an extreme point of the compact convex set $L := \{(x,t)|\ \min_{i \in I} v_i(x) \geq t,\ x \in C\}$. By the above mentioned trivial property, $(x^*, v(x^*))$ is a DH-point of L, i.e., there is no $x \in S(x^*)$ such that $\{i \in I|\ v_i(x) = v(x)\} \setminus \{i \in I|\ v_i(x^*) = v(x^*)\} \neq \emptyset$. Du-Hwang's result then follows by observing that $v_i(x) = v(x) \Leftrightarrow g_i(x) = g(x)$. □

4 Extreme point and DH-point — general case

The proof of Theorem 2 is based on the continuity of the function $x \mapsto \max_{y \in D \setminus M(x^*)} g(x,y)$. Since this continuity is obvious only when D is finite, the proof does not carry over to the case $|D| = +\infty$. To extend Theorem 2 to the case where D may not be finite we need a more involved proof.

In problem (GCP) assume now that the objective function $f(x)$ is quasiconcave and

(*) $g(x,y)$ is quasiconcave in x for every fixed y and continuous jointly in x and y on the product topology of $C \times D$.

Corollary 2 can be generalized as follows.

Theorem 3 *Under the stated assumptions, at least an optimal solution of (GCP) is a DH-point of its feasible set.*

Proof Let α be the optimal value of (GCP). With $\omega \in Y \setminus D$, define $z = (x,t),\ g(x,\omega) = f(x) - \alpha,\ D' = D \cup \{\omega\},\ \tilde{g}(x) = \max_{y \in D'} g(x,y),\ T = \max\{g(x,y)|\ x \in C, y \in D'\}$, and consider the problem $\min\{\tilde{g}(x)|\ x \in C\}$, or, equivalently,

$$\min\{t|\ \tilde{g}(x) := \max_{y \in D'} g(x,y) \leq t,\ (x,t) \in C \times [0,T]\}. \tag{Q}$$

By Theorem 1, there exists an extreme point x^* of K which is an optimal solution of (GCP). Clearly $z^* := (x^*, 0)$ is a feasible solution of (Q), while for any feasible solution $z = (x,t)$ of (Q) one has $t \geq 0$, so 0 is the optimal value, and $z^* = (x^*, 0)$ an optimal solution, of (Q).

Let

$$H := \{z = (x,t)|\ \max_{y \in D'} g(x,y) \leq t,\ (x,t) \in C \times [0,T]\}.$$

Define on H a relation $\preceq$ as follows:

$$z = (x,t), z' = (x',t') \in H,\ z \preceq z' \Leftrightarrow \{S(x') \subset S(x) \text{ and } M(z) \subset M(z')\},$$

where $S(x)$ is, as previously, the smallest face of C containing x, while $M(z) = \{y \in D'|\ g(x,y) = t\}$. For $(x,t) \in H$, we have $t \geq \max_{y \in D'} g(x,y)$, so $g(x,y) = t \Leftrightarrow$

$g(x, y) = \tilde{g}(x)$, hence $M(z) = \{y \in D' | \ g(x, y) = \tilde{g}(x)\}$. It is plain to verify that the relation $\preceq$ is reflexive and transitive, i.e., is a partial ordering on H. Let $Z^* := \{z \in H | \ z^* \preceq z\}$. For each sequence $z^* \preceq z^1 \preceq z^2 \preceq \cdots$ in Z^* there is a subsequence $z^{k_\nu} = (x^{k_\nu}, t^{k_\nu}),\ \nu = 1, 2, \ldots$ such that $z^{k_\nu} \to \bar{z} = (\bar{x}, \bar{t})$ with $\bar{x} \in S(x^*), (\bar{x}, \bar{t}) \in H$. Since for each fixed $h, x^k \in S(x^h)\ \forall k \geq h$, we have $\bar{x} \in S(x^h)\ \forall h$, but $\cap_h S(x^h)$ is a face of $S(x^k)$, hence $S(\bar{x}) \subset S(x^k)\ \forall k$. Also, for every k, since $M(z^k) \subset M(z^h)\ \forall h \geq k$, it follows that if $y \in M(z^k)$ then $g(x^h, y) = \tilde{g}(x^h)\ \forall h \geq k$, hence, by continuity, $g(\bar{x}, y) = \tilde{g}(\bar{x})$, i.e., $y \in M(\bar{x})$. This means $M(z^k) \subset M(\bar{z})\ \forall k$, and so any sequence $\{z^k\} \subset Z^*$ as described has an upper bound $\bar{z} \in Z^*$. By Zorn Lemma, there exists a maximal element $\hat{z}$ of the set Z^*, i.e., a point $\hat{z} = (\hat{x}, \hat{t}) \in H$ such that for every $z \in H : \hat{z} \preceq z$ entails $z \preceq \hat{z}$. Equivalently, there is no $z = (x, t) \in H$ such that

$$S(x) \subset S(\hat{x}),\ \ M(\hat{z}) \subset M(z),\ \ (S(\hat{x}) \setminus S(x)) \cup (M(z) \setminus M(\hat{z})) \neq \emptyset,$$

so $\hat{z} = (\hat{x}, \hat{t})$ is a DH-point of H. Hence, by Lemma 1 x^* is a DH-point of K.

We now contend that $\hat{z}$ is an optimal solution. Suppose the contrary, that $g_0(\hat{x}, 0) := f(\hat{x}) - \alpha > 0$, so that $\tilde{g}(\hat{x}) > 0$. Let x^λ be a point such that $x^* = (1 - \lambda)x^\lambda + \lambda\hat{x}$. Noting that $\hat{x} \in S(x^*) \setminus \{x^*\}$ we have $x^\lambda \in S(x^*)$ for all sufficiently small $\lambda > 0$, say $\lambda \in (0, \bar{\lambda})$. For every $y \in M(z^*) \subset M(\hat{z})$ we have $g(x^*, y) = 0,\ \ g(\hat{x}, y) = \tilde{g}(\hat{x}) > 0$, but by quasiconcavity of $g(., y) :\ 0 \geq \min\{g(x^\lambda, y), g(\hat{x}, y)\}$, hence $g(x^\lambda, y) \leq 0$. Furthermore, since x^* is an extreme point of K, one cannot have $x^\lambda \in K$ for any $\lambda \in (0, \bar{\lambda})$, so for each λ there exists $y^\lambda \in D' \setminus M(z^*)$ with $g(x^\lambda, y^\lambda) > \tilde{g}(x^\lambda)$. By passing to subsequences if necessary, we can assume that $x^\lambda \to x^*,\ \ y^\lambda \to y^*$ as $\lambda \to 0$. Clearly $g(x^*, y^*) = \tilde{g}(x^*) = 0$, so $y^* \in M(z^*) \subset M(\hat{z})$. This entails $g(\hat{x}, y^*) = \tilde{g}(\hat{x}) > 0 = g(x^*, y^*)$, and hence $g(\hat{x}, y^\lambda) > g(x^*, y^\lambda)$ for $\lambda > 0$ sufficiently small. On the other hand, noting that $y^\lambda \notin M(z^*)$ we have $g(x^*, y^\lambda) < 0 = g(x^*, y^*) < g(x^\lambda, y^\lambda)$. So $g(x^*, y^\lambda) < \min\{g(x^\lambda, y^\lambda), g(\hat{x}, y^\lambda)\}$, conflicting with the quasiconcavity of $g(., y^\lambda)$. Therefore, $\hat{x}$ is an optimal solution, and since it is a DH-point, this completes the proof of the theorem. □

Corollary 4 *Under assumption (*), the minimum of* $g(x) := \max_{y \in D} g(x, y)$ *over* C *is achieved at a DH-point.*

Proof According to Theorem 3 an optimal solution $(x^*, g(x^*))$ of the problem

$$\min\{t | \ \max_{y \in D} g(x, y) \leq t,\ \ x \in C\}$$

exists which is a DH-point of the set $\{(x, t) | \ \max_{y \in D} g(x, y) \leq t,\ \ x \in C\}$. Then x^* is an optimal solution of the problem $\min\{g(x) | \ x \in C\}$ and by Lemma 1, x^* is a DH-point. □

Remark 2 In [3] the original result of Du-Hwang is quoted in the following much weaker form (Theorem 1 in [3]): if every $g_i(x),\ i = 1, \ldots, m$, is a concave continuous function on $\mathbb{R}^n$, then the minimum of $g(x) := \max_{y \in D} g(x, y)$ over a polytope $C \subset \mathbb{R}^n$ is achieved at some point x^* satisfying the condition:

(DP) *there exists a face* Z *of* C *such that* $x^* \in Z$ *and* $M(x^*) := \{y | \ g(x^*, y) = g(x^*)\}$ *is maximal (w.r.t. inclusion) in the family* $M(x), x \in Z$.

It is obvious that a DH-point x^* always satisfies condition (DP), but the converse is not true, i.e., a point x^* satisfying condition (DP) may not be a DH-point. To see this, it suffices to consider the problem $\min\{g(x) | \ x \in C\}$, where $g(x)$ is a concave function and C is a polytope defined by the affine inequalities $\langle a^j, x\rangle \leq b_j,\ \ j = 1, \ldots, p$. In this case $|D| = 1$, so $M(x)$ is the same for all x, and any $x^* \in \text{ri}C$ satisfies condition (DP) (with $Z = C$) but is not a DH-point, because for any boundary point x of C we have $M(x) = M(x^*)$, while $S(x^*) \setminus S(x) \neq \emptyset$.

Thus, Du-Pardalos' theorem in [3] is a continuous version of Theorem 1 in [3]) but not exactly a continuous version of Du-Hwang's minimax theorem in [2]. In a subsequent work [4] a slightly different continuous version is formulated (Theorem 11.1.7 in [3]), where, however, it is not clear what is meant by DH-point when D may be infinite. From the context the most reasonable interpretation is that (DP) is now replaced by (DP)*:

$M(x^*)$ is maximal over $S(x^*)$, the smallest face of C containing x^*.

Though condition (DP)* is certainly stronger than (DP), again it does not imply that x^* is a DH-point in the sense originally introduced in [2] when $|D| < +\infty$. A counter example is furnished by the problem $\min\{f(x)|\ \langle a^j, x\rangle \geq b_j,\ j = 1, \ldots, m\}$ with a concave objective function $f(x)$ such that every point in a face $Z = \{x|\ \langle a^{j_0}, x\rangle = b_{j_0}\}$ is an optimal solution. Here also $|D| = 1$, $M(x)$ is constant for all x, and every point of Z satisfies (DP)* but only the extreme points of Z are DH-points. Besides, the proof sketched in [4] for Theorem 11.1.7 does not really work.

5 Conclusion

In this paper we have shown that the DH-property introduced in [2] for optimal solutions of problem (1) is in fact a special case of a property of the extreme points of a compact set K that achieve the minimum of a quasiconcave function over K. In the case considered by Du-Hwang [2] $K = \{x \in C|\ \max_{i=1,\ldots,m} g_i(x) \leq 0\}$, where C is a polytope and $g_i(x), i = 1, \ldots, m$, are concave continuous functions. In the general case $K = \{x \in C|\ \max_{y\in D} g_y(x, y) \leq 0\}$ where C is an arbitrary compact convex set, and $g(x, y)$, $y \in D$, are quasiconcave functions in x for every fixed $y \in D$ and continuous jointly in (x, y) in the product topology of $C \times D$.

A related open question of interest is whether there exists a DC representation for the function $\max_{y\in D} g(x, y)$. More specifically, does there exist for each $x^0 \in C$ a neighborhood $W(x^0)$ of x^0 together with a finite set $I(x^0) \subset D$ such that $\max_{y\in D} g(x, y) = \max_{y\in I(x^0)} g(x, y)\ \forall x \in I(x^0)$?

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